

# Semilinear fractional elliptic equations with measures in unbounded domain

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## Abstract

In this paper, we study the existence of nonnegative weak solutions to (E)  $(-\Delta)^\alpha u + h(u) = \nu$  in a general regular domain  $\Omega$ , which vanish in  $\mathbb{R}^N \setminus \Omega$ , where  $(-\Delta)^\alpha$  denotes the fractional Laplacian with  $\alpha \in (0, 1)$ ,  $\nu$  is a nonnegative Radon measure and  $h : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is a continuous nondecreasing function satisfying a subcritical integrability condition.

Furthermore, we analyze properties of weak solution  $u_k$  to (E) with  $\Omega = \mathbb{R}^N$ ,  $\nu = k\delta_0$  and  $h(s) = s^p$ , where  $k > 0$ ,  $p \in (0, \frac{N}{N-2\alpha})$  and  $\delta_0$  denotes Dirac mass at the origin. Finally, we show for  $p \in (0, 1 + \frac{2\alpha}{N}]$  that  $u_k \rightarrow \infty$  in  $\mathbb{R}^N$  as  $k \rightarrow \infty$ , and for  $p \in (1 + \frac{2\alpha}{N}, \frac{N}{N-2\alpha})$  that  $\lim_{k \rightarrow \infty} u_k(x) = c|x|^{-\frac{2\alpha}{p-1}}$  with  $c > 0$ , which is a classical solution of  $(-\Delta)^\alpha u + u^p = 0$  in  $\mathbb{R}^N \setminus \{0\}$ .

**Key words:** Fractional Laplacian, Radon measure, Dirac mass, Singularities.

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## 1 Introduction

Let  $\Omega$  be a regular domain (not necessary bounded) of  $\mathbb{R}^N$  ( $N \geq 2$ ),  $\alpha \in (0, 1)$  and  $d\omega(x) = \frac{dx}{1+|x|^{N+2\alpha}}$ . Denote by  $\mathfrak{M}^b(\Omega)$  the space of the Radon measures  $\nu$  in  $\Omega$  such that  $\|\nu\|_{\mathfrak{M}^b(\Omega)} := |\nu|(\Omega) < +\infty$  and by  $\mathfrak{M}_+^b(\Omega)$  the nonnegative cone. The purpose of this paper is to study the existence of nonnegative weak solutions to semilinear fractional elliptic equations

$$\begin{aligned} (-\Delta)^\alpha u + h(u) &= \nu & \text{in } \Omega, \\ u &= 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{aligned} \quad (1.1)$$

where  $h : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is a continuous nondecreasing function and  $(-\Delta)^\alpha$  denotes the fractional Laplacian of exponent  $\alpha$  defined by

$$(-\Delta)^\alpha u(x) = \lim_{\epsilon \rightarrow 0^+} (-\Delta)_\epsilon^\alpha u(x),$$

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where for  $\epsilon > 0$ ,

$$(-\Delta)_\epsilon^\alpha u(x) = - \int_{\mathbb{R}^N} \frac{u(z) - u(x)}{|z - x|^{N+2\alpha}} \chi_\epsilon(|x - z|) dz$$

and

$$\chi_\epsilon(t) = \begin{cases} 0, & \text{if } t \in [0, \epsilon], \\ 1, & \text{if } t > \epsilon. \end{cases}$$

In the pioneering work [5] (also see [3]), Brezis studied the existence of weak solutions to second order elliptic problem

$$\begin{aligned} -\Delta u + h(u) &= \nu & \text{in } \Omega, \\ u &= 0 & \text{on } \partial\Omega, \end{aligned} \tag{1.2}$$

where  $\Omega$  is a bounded  $C^2$  domain in  $\mathbb{R}^N$  ( $N \geq 3$ ),  $\nu$  is a bounded Radon measure in  $\Omega$ , and the function  $h : \mathbb{R} \rightarrow \mathbb{R}$  is nondecreasing, positive on  $(0, +\infty)$  and satisfies the subcritical assumption:

$$\int_1^{+\infty} (h(s) - h(-s)) s^{-2\frac{N-1}{N-2}} ds < +\infty.$$

In particular case that  $0 \in \Omega$ ,  $h(s) = s^q$  and  $\nu = k\delta_0$  with  $k > 0$ , Brezis et al showed that (1.2) admits a unique weak solution  $v_k$  for  $1 < q < N/(N-2)$ , while no solution exists if  $q \geq N/(N-2)$ . Later on, Véron in [29] proved that if  $1 < q < N/(N-2)$ , the limit of  $v_k$  is a strong singular solution of

$$\begin{aligned} -\Delta u + u^q &= 0 & \text{in } \Omega \setminus \{0\}, \\ u &= 0 & \text{on } \partial\Omega. \end{aligned} \tag{1.3}$$

After that, Brezis-Véron in [8] found that the problem (1.3) admits only the zero solution if  $q \geq N/(N-2)$ . Much advances in the study of semilinear second order elliptic equations involving measures see references [2, 6, 22, 23].

During the last years, there has also been a renewed and increasing interest in the study of linear and nonlinear integro-differential operators, especially, the fractional Laplacian, motivated by various applications in physics and by important links on the theory of Lévy processes, refer to [9, 11, 10, 12, 13, 14, 16, 17, 19, 25, 26]. In a recent work, Karisen-Petitta-Ulusoy in [20] used the duality approach to study the fractional elliptic equation

$$(-\Delta)^\alpha v = \mu \quad \text{in } \mathbb{R}^N,$$

where  $\mu$  is a Radon measure with compact support. More recently, Chen-Véron in [17] studied the semilinear fractional elliptic problem (1.1) when  $\Omega$  is an open bounded regular set in  $\mathbb{R}^N$  and  $\nu$  is a Radon measure such

that  $\int_{\Omega} d^{\beta} d|\nu| < +\infty$  with  $\beta \in [0, \alpha]$  and  $d(x) = \text{dist}(x, \partial\Omega)$ . The existence and uniqueness of weak solution are obtained when the function  $h$  is nondecreasing and satisfies

$$\int_1^{+\infty} (h(s) - h(-s)) s^{-1-k_{\alpha,\beta}} ds < +\infty, \quad (1.4)$$

where

$$k_{\alpha,\beta} = \begin{cases} \frac{N}{N-2\alpha}, & \text{if } \beta \in [0, \frac{N-2\alpha}{N}\alpha], \\ \frac{N+\alpha}{N-2\alpha+\beta}, & \text{if } \beta \in (\frac{N-2\alpha}{N}\alpha, \alpha]. \end{cases} \quad (1.5)$$

Motivated by these results and in view of the non-local character of the fractional Laplacian we are interested in the existence of weak solutions to problem (1.1) when  $\Omega$  is a general regular domain, including  $\Omega = \mathbb{R}^N$ . Before stating our main results in this paper, we introduce the definition of weak solution to (1.1).

**Definition 1.1** *A function  $u \in L^1(\mathbb{R}^N, d\omega)$  is a weak solution of (1.1) if  $h(u) \in L^1(\mathbb{R}^N, d\omega)$  and*

$$\int_{\Omega} [u(-\Delta)^{\alpha}\xi + h(u)\xi] dx = \int_{\Omega} \xi d\nu, \quad \forall \xi \in \mathbb{X}_{\Omega}, \quad (1.6)$$

where  $\mathbb{X}_{\Omega} \subset C(\mathbb{R}^N)$  is the space of functions  $\xi$  satisfying:

- (i) the support of  $\xi$  is a compact set in  $\bar{\Omega}$ ;
- (ii)  $(-\Delta)^{\alpha}\xi(x)$  exists for any  $x \in \Omega$  and there exists  $C > 0$  such that

$$|(-\Delta)^{\alpha}\xi(x)| \leq \frac{C}{1+|x|^{N+2\alpha}}, \quad \forall x \in \Omega;$$

- (iii) there exist  $\varphi \in L^1(\Omega, \rho^{\alpha} dx)$  and  $\epsilon_0 > 0$  such that  $|(-\Delta)^{\alpha}_\epsilon \xi| \leq \varphi$  a.e. in  $\Omega$ , for all  $\epsilon \in (0, \epsilon_0]$ , here  $\rho(x) = \min\{1, \text{dist}(x, \partial\Omega)\}$  if  $\Omega \neq \mathbb{R}^N$  or  $\rho \equiv 1$  if  $\Omega = \mathbb{R}^N$ .

We notice that  $\mathbb{X}_{\Omega}$  coincides with the test function space  $\mathbb{X}_{\alpha}$  if  $\Omega$  is bounded, see [17, Definition 1.1]. Moreover, the test function space  $\mathbb{X}_{\Omega}$  is used as  $C_0^{1,1}(\Omega)$  if  $\Omega$  is bounded and  $\alpha = 1$ , see [30]. We denote by  $G_{\Omega}$  the Green kernel of  $(-\Delta)^{\alpha}$  in  $\Omega \times \Omega$  and by  $\mathbb{G}_{\Omega}[\cdot]$  the Green operator defined as

$$\mathbb{G}_{\Omega}[\nu](x) = \int_{\Omega} G_{\Omega}(x, y) d\nu(y), \quad \forall \nu \in \mathfrak{M}^b(\Omega).$$

Now we are ready to state our first theorem on the existence of weak solutions for problem (1.1).

**Theorem 1.1** Assume that  $\alpha \in (0, 1)$ ,  $\Omega$  is a regular domain in  $\mathbb{R}^N$  ( $N \geq 2$ ) and  $h : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is a continuous nondecreasing function satisfying

$$\int_1^{+\infty} h(s) s^{-1-\frac{N}{N-2\alpha}} ds < +\infty. \quad (1.7)$$

Then for any  $\nu \in \mathfrak{M}_+^b(\Omega)$ , problem (1.1) admits a weak solution  $u_\nu$  such that

$$0 \leq u_\nu \leq \mathbb{G}_\Omega[\nu] \quad \text{a.e. in } \Omega. \quad (1.8)$$

If  $\nu$  is a nonnegative bounded Radon measure, (1.4) with  $\beta = 0$  and (1.7) have the same critical value  $\frac{N}{N-2\alpha}$ .

In the case that  $\Omega$  is bounded, the authors of [17] took a sequence of  $C^1$  functions  $\{\nu_n\}$  converging to  $\nu$  in the weak star sense, then they considered the solutions  $u_n$  of (1.1) replacing  $\nu$  by  $\nu_n$ . By the compact imbedding theorem, they showed that the limit of  $\{u_n\}$  exists, up to subsequence. While for the case that  $\Omega$  is unbounded, the difficulty is that Sobolev imbedding may not be compact. To overcome the difficulty, we truncate the measure  $\nu$  by  $\nu \chi_{B_R(0)}$  and use the increasing monotonicity of corresponding solutions  $\{u_R\}$  of solutions to (1.1) in related bounded domains. Taking the limit as  $R \rightarrow \infty$ , we achieve the desired weak solution.

The second purpose in this paper is to study properties of weak solution to problem (1.1) when  $\Omega = \mathbb{R}^N$ ,  $h(u) = u^p$  and  $\nu = k\delta_0$ , that is,

$$\begin{aligned} (-\Delta)^\alpha u + u^p &= k\delta_0 \quad \text{in } \mathbb{R}^N, \\ \lim_{|x| \rightarrow +\infty} u(x) &= 0, \end{aligned} \quad (1.9)$$

where  $p \in (0, \frac{N}{N-2\alpha})$ ,  $k > 0$  and  $\delta_0$  denotes the Dirac mass at the origin.

**Theorem 1.2** Assume that  $\alpha \in (0, 1)$  and  $p \in (0, \frac{N}{N-2\alpha})$ . Then for any  $k > 0$ , problem (1.9) admits a unique weak solution  $u_k$  such that

$$\lim_{x \rightarrow 0} u_k(x) |x|^{N-2\alpha} = c_1 k, \quad (1.10)$$

where  $c_1 > 0$ . Moreover,

(i)  $\{u_k\}_{k \in (0, \infty)}$  are classical solutions of

$$(-\Delta)^\alpha u + u^p = 0 \quad \text{in } \mathbb{R}^N \setminus \{0\}; \quad (1.11)$$

(ii) the mapping:  $k \mapsto u_k$  is increasing.

We consider the asymptotic behavior of  $u_1$  at  $\infty$  when  $p \in (1, \frac{N}{N-2\alpha})$ .

**Theorem 1.3** Assume that  $\alpha \in (0, 1)$  and  $u_1$  is the solution of (1.9) with  $k = 1$ . Then there exist  $c_2 > 1$  and  $R > 2$  such that for  $|x| \geq R$ ,

(i) if  $p \in (1, 1 + \frac{2\alpha}{N})$ ,

$$\frac{1}{c_2}|x|^{-\frac{N+2\alpha}{p}} \leq u_1(x) \leq c_2|x|^{-\frac{N+2\alpha}{p}}; \quad (1.12)$$

(ii) if  $p = 1 + \frac{2\alpha}{N}$ ,

$$\frac{1}{c_2}|x|^{-N} \log^{\frac{N}{2\alpha}}(|x|) \leq u_1(x) \leq c_2|x|^{-N} \log^{\frac{N}{2\alpha}}(|x|); \quad (1.13)$$

(iii) if  $p \in (1 + \frac{2\alpha}{N}, \frac{N}{N-2\alpha})$ ,

$$\frac{1}{c_2}|x|^{-\frac{2\alpha}{p-1}} \leq u_1(x) \leq c_2|x|^{-\frac{2\alpha}{p-1}}. \quad (1.14)$$

According to Theorem 1.3, we know that the decaying power of  $u_1$  shifts at the point  $p = 1 + \frac{2\alpha}{N}$ ; while for  $\alpha = 1$  and  $p \in (1, \frac{N}{N-2})$ , the weak solution of (1.9) decays as  $|x|^{-\frac{2}{p-1}}$ .

From now on, we denote that  $u_k$  is the weak solution of (1.9). Since the mapping:  $k \mapsto u_k$  is increasing by Theorem 1.2 and then we can denote that

$$u_\infty(x) = \lim_{k \rightarrow \infty} u_k(x), \quad x \in \mathbb{R}^N. \quad (1.15)$$

Here we note that  $u_\infty(x) \in \mathbb{R}_+ \cup \{+\infty\}$  for any  $x \in \mathbb{R}^N$ . Now we state properties of  $u_\infty$ .

**Theorem 1.4** Assume that  $\alpha \in (0, 1)$ ,  $p \in (0, \frac{N}{N-2\alpha})$  and  $u_\infty$  is given by (1.15). Then

(i) if  $p \in (0, 1 + \frac{2\alpha}{N}]$ , then  $u_\infty(x) = \infty$ ,  $\forall x \in \mathbb{R}^N$ ;

(ii) if  $p \in (1 + \frac{2\alpha}{N}, \frac{N}{N-2\alpha})$ , then  $u_\infty$  is a classical solution of (1.11) and there exists  $c_3 > 0$  such that

$$u_\infty(x) = c_3|x|^{-\frac{2\alpha}{p-1}}, \quad \forall x \in \mathbb{R}^N \setminus \{0\}.$$

In the proof of Theorem 1.4, we make use of the self-similar property of  $u_\infty$ .

Analogue results of Theorem 1.4 in bounded domain  $\Omega$  were obtained [17, 18]. Precisely, they showed that there exists a unique weak solution  $u_{k,\Omega}$  to semilinear fractional elliptic problem

$$\begin{aligned} (-\Delta)^\alpha u + u^p &= k\delta_0 & \text{in } \Omega, \\ u &= 0 & \text{in } \Omega^c, \end{aligned} \quad (1.16)$$

where  $k > 0$ ,  $0 \in \Omega$  and  $p \in (0, \frac{N}{N-2\alpha})$ . Moreover,  
(i) the mapping  $k \mapsto u_{k,\Omega}$  is increasing;  
(ii) for  $p \in (0, \min\{1 + \frac{2\alpha}{N}, \frac{N}{2\alpha}\})$ ,  $u_{\infty,\Omega} = \infty$  in  $\Omega$ , where

$$u_{\infty,\Omega} = \lim_{k \rightarrow \infty} u_{k,\Omega} \quad \text{in } \mathbb{R}^N;$$

(iii) for  $p \in (1 + \frac{2\alpha}{N}, \frac{N}{N-2\alpha})$ ,  $u_{\infty,\Omega}$  is a classical solution of

$$\begin{aligned} (-\Delta)^\alpha u + u^p &= 0 & \text{in } \Omega \setminus \{0\}, \\ u &= 0 & \text{in } \Omega^c. \end{aligned} \tag{1.17}$$

Finally, we discuss properties of weak solution  $u_{k,\Omega}$  of (1.16) when  $\Omega$  is an unbounded regular domain including the origin. The result is stated as follows.

**Theorem 1.5** *Assume that  $\alpha \in (0, 1)$ ,  $\Omega$  is an unbounded regular domain of  $\mathbb{R}^N$  ( $N \geq 2$ ) including the origin,  $p \in (0, \frac{N}{N-2\alpha})$  and  $u_k$  is given by Theorem 1.2. Then*

(i) *for any  $k > 0$ , (1.16) admits a unique weak solution  $u_{k,\Omega}$  such that*

$$u_k - m_{k,\Omega} \leq u_{k,\Omega} \leq u_k \quad \text{in } \Omega$$

*and the mapping  $k \mapsto u_{k,\Omega}$  is increasing, where  $m_{k,\Omega} = \sup_{x \in \Omega^c} u_k(x)$ ;*

(ii) *for  $p \in (0, \min\{1 + \frac{2\alpha}{N}, \frac{N}{2\alpha}\})$ ,  $u_{\infty,\Omega} = \infty$  in  $\Omega$ , where*

$$u_{\infty,\Omega} = \lim_{k \rightarrow \infty} u_{k,\Omega} \quad \text{in } \mathbb{R}^N;$$

(iii) *for  $p \in (1 + \frac{2\alpha}{N}, \frac{N}{N-2\alpha})$ ,  $u_{\infty,\Omega}$  is a classical solution of (1.17) such that*

$$u_\infty - m_{\infty,\Omega} \leq u_{\infty,\Omega} \leq u_\infty \quad \text{in } \Omega,$$

*where  $u_\infty$  is defined by (1.15) and  $m_{\infty,\Omega} = \sup_{x \in \Omega^c} u_\infty(x)$ .*

The paper is organized as follows. In Section 2 we list some properties of Marcinkiewicz spaces and establish the inequality

$$\|\mathbb{G}_{\mathbb{R}^N}[\nu]\|_{M^{\frac{N}{N-2\alpha}}(\mathbb{R}^N, d\omega)} \leq c_5 \|\nu\|_{\mathfrak{M}^b(\mathbb{R}^N)}, \tag{1.18}$$

which is used to obtain that  $h(\mathbb{G}_{\mathbb{R}^N}[\nu]) \in L^1(\mathbb{R}^N, d\omega)$ . In Section 3, we prove Theorem 1.1. The proofs of Theorem 1.2 and Theorem 1.3 are addressed in Section 4. Finally, we give the proofs of Theorem 1.4 and Theorem 1.5 in Section 5.

## 2 Preliminary

The purpose of this section is to introduce some preliminaries and prove Marcinkiewicz type estimate.

## 2.1 Marcinkiewicz type estimate

In this subsection, we recall the definition of Marcinkiewicz space and prove Marcinkiewicz type estimate.

**Definition 2.1** Let  $\Theta \subset \mathbb{R}^N$  be a domain and  $\mu$  be a positive Borel measure in  $\Theta$ . For  $\kappa > 1$ ,  $\kappa' = \kappa/(\kappa - 1)$  and  $u \in L_{loc}^1(\Theta, d\mu)$ , we set

$$\|u\|_{M^\kappa(\Theta, d\mu)} = \inf \left\{ c \in [0, \infty] : \int_E |u| d\mu \leq c \left( \int_E d\mu \right)^{\frac{1}{\kappa'}}, \forall E \subset \Theta, E \text{ Borel} \right\} \quad (2.1)$$

and

$$M^\kappa(\Theta, d\mu) = \{u \in L_{loc}^1(\Theta, d\mu) : \|u\|_{M^\kappa(\Theta, d\mu)} < \infty\}. \quad (2.2)$$

$M^\kappa(\Theta, d\mu)$  is called the Marcinkiewicz space of exponent  $\kappa$ , or weak  $L^\kappa$ -space and  $\|\cdot\|_{M^\kappa(\Theta, d\mu)}$  is a quasi-norm.

**Proposition 2.1** [4] Assume that  $1 \leq q < \kappa < \infty$  and  $u \in L_{loc}^1(\Theta, d\mu)$ . Then there exists  $c_4 > 0$  dependent of  $q, \kappa$  such that

$$\int_E |u|^q d\mu \leq c_4 \|u\|_{M^\kappa(\Theta, d\mu)} \left( \int_E d\mu \right)^{1-q/\kappa},$$

for any Borel set  $E$  of  $\Theta$ .

Now we are ready to state Marcinkiewicz type estimate as follows.

**Proposition 2.2** Let  $\nu \in \mathfrak{M}^b(\mathbb{R}^N)$ , then there exists  $c_5 > 0$  such that

$$\|\mathbb{G}_{\mathbb{R}^N}[\|\nu\|]\|_{M^{p_\alpha^*}(\mathbb{R}^N, d\omega)} \leq c_5 \|\nu\|_{\mathfrak{M}^b(\mathbb{R}^N)}, \quad (2.3)$$

where  $d\omega(x) = \frac{dx}{1+|x|^{N+2\alpha}}$  and  $p_\alpha^* = \frac{N}{N-2\alpha}$ .

*Proof.* For  $\lambda > 0$  and  $y \in \mathbb{R}^N$ , we set

$$A_\lambda(y) = \{x \in \mathbb{R}^N \setminus \{y\} : G_{\mathbb{R}^N}(x, y) > \lambda\}, \quad m_\lambda(y) = \int_{A_\lambda(y)} d\omega.$$

We observe that there exists a positive constant  $c_{N,\alpha}$  such that

$$G_{\mathbb{R}^N}(x, y) = \frac{c_{N,\alpha}}{|x - y|^{N-2\alpha}}, \quad (x, y) \in \mathbb{R}^N \times \mathbb{R}^N, \quad x \neq y,$$

which implies that

$$A_\lambda(y) \subset \left\{ x \in \mathbb{R}^N : |x - y| \leq c_{N,\alpha} \lambda^{-\frac{1}{N-2\alpha}} \right\}. \quad (2.4)$$

As a consequence,

$$m_\lambda(y) = \int_{A_\lambda(y)} \frac{dx}{1 + |x|^{N+2\alpha}} \leq |A_\lambda(y)| \leq c_6 \lambda^{-p_\alpha^*}, \quad (2.5)$$

where  $c_6 > 0$  independent of  $y$  and  $\lambda$  and  $p_\alpha^* = \frac{N}{N-2\alpha}$ .

Let  $E \subset \mathbb{R}^N$  be a Borel set and  $\lambda > 0$ , then

$$\int_E G_{\mathbb{R}^N}(x, y) d\omega(x) \leq \int_{A_\lambda(y)} G_{\mathbb{R}^N}(x, y) d\omega(x) + \lambda \int_E d\omega.$$

By (2.5), we have that

$$\int_{A_\lambda(y)} G_{\mathbb{R}^N}(x, y) d\omega(x) = \lambda m_\lambda(y) + \int_\lambda^\infty m_s(y) ds \leq c_7 \lambda^{1-p_\alpha^*},$$

for some  $c_7 > 0$ , then it results that

$$\int_E G_{\mathbb{R}^N}(x, y) d\omega(x) \leq c_7 \lambda^{1-p_\alpha^*} + \lambda \int_E d\omega.$$

Choosing  $\lambda = (\int_E d\omega)^{-\frac{1}{p_\alpha^*}}$ , we obtain

$$\int_E G_{\mathbb{R}^N}(x, y) d\omega(x) \leq (c_7 + 1) \left( \int_E d\omega \right)^{\frac{p_\alpha^*-1}{p_\alpha^*}}, \quad \forall y \in \mathbb{R}^N.$$

Therefore,

$$\begin{aligned} \int_E \mathbb{G}_{\mathbb{R}^N}[|\nu|](x) d\omega(x) &= \int_{\mathbb{R}^N} \int_E G_{\mathbb{R}^N}(x, y) d\omega(x) d|\nu(y)| \\ &\leq (c_7 + 1) \int_{\mathbb{R}^N} d|\nu(y)| \left( \int_E d\omega \right)^{\frac{p_\alpha^*-1}{p_\alpha^*}} \\ &\leq (c_7 + 1) \|\nu\|_{\mathfrak{M}^b(\mathbb{R}^N)} \left( \int_E d\omega \right)^{\frac{p_\alpha^*-1}{p_\alpha^*}}. \end{aligned} \quad (2.6)$$

As a consequence,

$$\|\mathbb{G}_{\mathbb{R}^N}[|\nu|]\|_{M^{p_\alpha^*}(\mathbb{R}^N, d\omega)} \leq c_5 \|\nu\|_{\mathfrak{M}^b(\mathbb{R}^N)},$$

which ends the proof.  $\square$

Now we use Marcinkiewicz type estimate to prove the following lemma, which is the key-stone in the proof of Theorem 1.1.

**Lemma 2.1** *Assume that  $\nu \in \mathfrak{M}_+^b(\mathbb{R}^N)$  and  $h : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is a continuous nondecreasing function satisfying (1.7). Then*

$$\mathbb{G}_{\mathbb{R}^N}[\nu], h(\mathbb{G}_{\mathbb{R}^N}[\nu]) \in L^1(\mathbb{R}^N, d\omega).$$



**Proof.** On the one hand, using Fubini's lemma, we have that

$$\begin{aligned}
\|\mathbb{G}_{\mathbb{R}^N}[\nu]\|_{L^1(\mathbb{R}^N, d\omega)} &= c_{N,\alpha} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{1}{1+|x|^{N+2\alpha}} \frac{1}{|x-y|^{N-2\alpha}} d\nu(y) dx \\
&= c_{N,\alpha} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{1}{1+|x|^{N+2\alpha}} \frac{1}{|x-y|^{N-2\alpha}} dx d\nu(y) \\
&\leq c_{N,\alpha} \int_{\mathbb{R}^N} [\int_{B_1(y)} \frac{1}{|x-y|^{N-2\alpha}} dx + \int_{B_1^c(y)} \frac{1}{1+|x|^{N+2\alpha}} dx] d\nu(y) \\
&< +\infty,
\end{aligned}$$

that is,  $\mathbb{G}_{\mathbb{R}^N}[\nu] \in L^1(\mathbb{R}^N, d\omega)$ .

On the other hand, let  $S_\lambda = \{x \in B_R(0) : \mathbb{G}_{\mathbb{R}^N}[\nu](x) > \lambda\}$  and  $g(\lambda) = \int_{S_\lambda} d\omega$ , where  $\lambda \geq 1$ . We observe that

$$\begin{aligned}
\int_{\mathbb{R}^N} h(\mathbb{G}_{\mathbb{R}^N}[\nu]) d\omega &= \int_{S_\lambda^c} h(\mathbb{G}_{\mathbb{R}^N}[\nu]) d\omega + \int_{S_\lambda} h(\mathbb{G}_{\mathbb{R}^N}[\nu]) d\omega \\
&\leq h(\lambda) \int_{\mathbb{R}^N} d\omega + \int_{S_\lambda} h(\mathbb{G}_{\mathbb{R}^N}[\nu]) d\omega \\
&= h(\lambda) \int_{\mathbb{R}^N} d\omega + h(\lambda) g(\lambda) + \int_\lambda^\infty g(s) dh(s).
\end{aligned} \tag{2.7}$$

Since

$$\int_\lambda^\infty g(s) dh(s) = \lim_{T \rightarrow \infty} \int_\lambda^T g(s) dh(s)$$

and  $\mathbb{G}_{\mathbb{R}^N}[\nu] \in M^{p_\alpha^*}(\mathbb{R}^N, d\omega)$ , it derives from Proposition 2.2 and Proposition 2.1 with  $q = 1$ ,  $\kappa = p_\alpha^*$ ,  $E = S_\lambda$  and  $d\mu = d\omega$  that  $g(s) \leq c_8 s^{-p_\alpha^*}$  and for  $T > \lambda$ ,

$$\begin{aligned}
h(\lambda)g(\lambda) + \int_\lambda^T g(s) dh(s) &\leq c_8 \lambda^{-p_\alpha^*} h(\lambda) + c_8 \int_\lambda^T s^{-p_\alpha^*} dh(s) \\
&\leq c_8 T^{-p_\alpha^*} h(T) + c_8 p_\alpha^* \int_\lambda^T s^{-1-p_\alpha^*} h(s) ds,
\end{aligned}$$

where  $c_8 > 0$ . By the nondecreasing monotonicity of  $h$ , we have that

$$\begin{aligned}
T^{-p_\alpha^*} h(T) &= 2^{1+p_\alpha^*} h(T) (2T)^{-1-p_\alpha^*} \int_T^{2T} dt \\
&\leq 2^{1+p_\alpha^*} \int_T^{2T} h(t) t^{-1-p_\alpha^*} dt,
\end{aligned}$$

then it infers by (1.7) that that

$$\lim_{T \rightarrow \infty} T^{-p_\alpha^*} h(T) = 0.$$

Therefore, by (1.7) and we take  $\lambda = 1$ ,

$$\int_{\mathbb{R}^N} h(\mathbb{G}_{\mathbb{R}^N}[\nu]) d\omega \leq h(1) \int_{\mathbb{R}^N} d\omega + c_8 p_\alpha^* \int_1^\infty s^{-1-p_\alpha^*} h(s) ds < +\infty,$$

i.e.  $h(\mathbb{G}_{\mathbb{R}^N}[\nu]) \in L^1(\mathbb{R}^N, d\omega)$ . We complete the proof.  $\square$

## 2.2 Basic results

This subsection is devoted to present some basic results and Comparison Principle, which are key tools in the analysis. We start it by recalling the existence of weak solution of (1.1) when  $\Omega$  is a bounded  $C^2$  domain.

**Proposition 2.3** [17, Theorem 1.1] *Assume that  $\mathcal{O}$  is a bounded  $C^2$  domain in  $\mathbb{R}^N$ ,  $\mu \in \mathfrak{M}_+^b(\mathcal{O})$  and  $h : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is a continuous nondecreasing function satisfying (1.7). Then problem*

$$\begin{aligned} (-\Delta)^\alpha u + h(u) &= \mu & \text{in } \mathcal{O}, \\ u &= 0 & \text{in } \mathbb{R}^N \setminus \mathcal{O} \end{aligned} \quad (2.8)$$

*admits a unique weak solution  $v_\mu$  such that*

$$0 \leq v_\mu \leq \mathbb{G}_{\mathcal{O}}[\mu] \quad \text{a.e. in } \mathcal{O}. \quad (2.9)$$

*Moreover, the mapping  $\mu \mapsto v_\mu$  is increasing.*

Next we recall the Comparison Principle from [12].

**Lemma 2.2** [12, Theorem 2.3] *Suppose that  $O$  is a bounded domain of  $\mathbb{R}^N$ ,  $p > 0$ , the functions  $u_1, u_2$  are continuous in  $\bar{O}$  and satisfy*

$$(-\Delta)^\alpha u_1 + |u_1|^{p-1}u_1 \geq 0 \text{ in } O \quad \text{and} \quad (-\Delta)^\alpha u_2 + |u_2|^{p-1}u_2 \leq 0 \text{ in } O.$$

*If  $u_1 \geq u_2$  a.e. in  $O^c$ , then  $u_1 \geq u_2$  in  $O$ .*

By the Comparison Principle, we have the following result:

**Lemma 2.3** *Assume that  $f \in C^1(\mathbb{R}^N)$  is a nonnegative function,  $h$  is a continuous and nondecreasing function and  $\mathcal{O}_1, \mathcal{O}_2$  are bounded  $C^2$  domain such that  $\mathcal{O}_1 \subset \mathcal{O}_2$ . Let  $w_1$  and  $w_2$  be the solutions of (2.8) with  $\mu = f$  in  $\mathcal{O} = \mathcal{O}_1$  and  $\mu = f$  in  $\mathcal{O} = \mathcal{O}_2$ , respectively. Then*

$$w_1 \leq w_2 \quad \text{in } \mathbb{R}^N.$$

**Proof.** Since  $\mathcal{O}_1 \subset \mathcal{O}_2$  and  $f \geq 0$ , it follows by Lemma 2.2 that  $w_2 \geq 0$  in  $\mathcal{O}_2$ . Suppose on the contrary that

$$\min_{x \in \mathbb{R}^N} (w_2 - w_1)(x) < 0,$$

there would exist  $x_0 \in \mathcal{O}_1$  such that

$$(w_2 - w_1)(x_0) = \min_{x \in \mathbb{R}^N} (w_2 - w_1)(x).$$

Then

$$(-\Delta)^\alpha (w_2 - w_1)(x_0) = -\lim_{\epsilon \rightarrow 0^+} \int_{\mathbb{R}^N \setminus B_\epsilon(x_0)} \frac{(w_2 - w_1)(z) - (w_2 - w_1)(x_0)}{|z - x_0|^{N+2\alpha}} dz < 0$$

and  $h(w_2(x_0)) \leq h(w_1(x_0))$ , which implies a contradiction since  $w_1$  and  $w_2$  satisfy  $(-\Delta)^\alpha u + h(u) = f(x_0)$ . The proof is completed.  $\square$

**Lemma 2.4** Suppose that  $p > 0$ ,  $\mathcal{O}$  is a bounded  $C^2$  domain in  $\mathbb{R}^N$ ,  $g \in L^1(\mathcal{O}^c, d\omega)$  is  $C^2$  in  $\{z \in \mathcal{O}^c, \text{dist}(z, \partial\mathcal{O}) \leq \delta\}$  with  $\delta > 0$ . Then there exists a unique classical solution  $u$  of

$$\begin{cases} (-\Delta)^\alpha u(x) + |u|^{p-1}u(x) = 0, & x \in \mathcal{O}, \\ u(x) = g(x), & x \in \mathcal{O}^c. \end{cases} \quad (2.10)$$

**Proof.** For the existence of classical solutions, we refer to Theorem 2.5 in [12]. The uniqueness follows by Lemma 2.2.  $\square$

### 3 Existence of weak solutions

In this section, we show the existence of solutions of problem (1.1), that is, we will prove Theorem 1.1. We first give an auxiliary lemma as follows.

**Lemma 3.1** Assume that  $\mathcal{O}$  is a bounded  $C^2$  domain in  $\mathbb{R}^N$  and  $\eta \in C(\mathbb{R}^N)$  with support in  $\bar{\mathcal{O}}$ . Then there exists  $c_9 > 0$  such that

$$|(-\Delta)^\alpha \eta(x)| \leq \frac{c_9 \|\eta\|_{L^\infty(\mathcal{O})}}{1 + |x|^{N+2\alpha}}, \quad x \in \mathbb{R}^N \setminus \mathcal{O}_d, \quad (3.1)$$

where  $\mathcal{O}_d = \{x \in \mathbb{R}^N : \text{dist}(\mathcal{O}, x) \leq d\}$ .

**Proof.** For  $x \in \mathcal{O}_d^c$  and  $y \in \mathcal{O}$ , there exists  $c_{10} > 1$  such that

$$c_{10}^{-1}(1 + |x|^{N+2\alpha}) \leq |x - y|^{N+2\alpha} \leq c_{10}(1 + |x|^{N+2\alpha}).$$

By the fact that

$$(-\Delta)^\alpha \eta(x) = - \int_{\mathcal{O}} \frac{\eta(y)}{|x - y|^{N+2\alpha}} dy, \quad x \in \mathcal{O}_d^c,$$

we assert (3.1) holds, which ends the proof.  $\square$

Now we are in the position to prove Theorem 1.1.

**Proof of Theorem 1.1.** Let  $\{\mathcal{O}_n\}_{n \in \mathbb{N}}$  be a sequence of  $C^2$  domains in  $\mathbb{R}^N$  such that

$$\Omega \cap B_n(0) \subset \{\mathcal{O}_n\} \subset \Omega \cap B_{n+1}(0).$$

For  $\nu \in \mathfrak{M}_+^b(\Omega)$  and  $n \in \mathbb{N}$ , we denote  $\nu_n = \nu \chi_n$ , where  $\chi_n$  is the characteristic function in  $\mathcal{O}_n$ . By Proposition 2.3, problem (2.8) with  $\mathcal{O} = \mathcal{O}_n$  and  $\mu = \nu_n$  admits a unique weak solution, denoting it by  $v_n$ . We divide the proof in following steps.

*Step 1.* We claim that  $v_n \leq v_{n+1}$  a.e. in  $\mathbb{R}^N$ . In fact, let  $\tilde{v}_{n+1}$  be the solution of (2.8) with  $\mathcal{O} = \mathcal{O}_{n+1}$  and  $\mu = \nu_n$ . By Proposition 2.3,

$$\tilde{v}_{n+1} \leq v_{n+1} \quad \text{a.e. in } \mathbb{R}^N. \quad (3.2)$$

Choosing a sequence nonnegative functions  $\{f_{n,m}\}_{m \in \mathbb{N}} \subset C_0^1(\mathcal{O}_n)$  such that  $f_{n,m} \rightarrow \nu_n$  as  $m \rightarrow \infty$  in the distribution sense, we make zero extension of  $f_{n,m}$  into  $C_0^1(\mathcal{O}_{n+1})$  and denote the extension by  $\tilde{f}_{n,m}$ . Let  $v_{n,m}$  and  $\tilde{v}_{n+1,m}$  be solutions of (2.8) with  $\mu = f_{n,m}$ ,  $\mathcal{O} = \mathcal{O}_n$  and  $\mu = \tilde{f}_{n,m}$ ,  $\mathcal{O} = \mathcal{O}_{n+1}$ , respectively. Lemma 2.3 implies that

$$v_{n,m} \leq \tilde{v}_{n+1,m} \quad \text{in } \mathbb{R}^N.$$

Together with the facts that  $v_{n,m} \rightarrow v_n$  a.e. in  $\mathbb{R}^N$  and  $\tilde{v}_{n+1,m} \rightarrow \tilde{v}_{n+1}$  a.e. in  $\mathbb{R}^N$  as  $m \rightarrow \infty$ , we obtain that

$$v_n \leq \tilde{v}_{n+1} \quad \text{a.e. in } \mathbb{R}^N. \quad (3.3)$$

It follows by (3.2) and (3.3) that for any  $n \in \mathbb{N}$ ,

$$v_n \leq v_{n+1} \quad \text{a.e. in } \mathbb{R}^N. \quad (3.4)$$

*Step 2. Uniform bounds of  $\{v_n\}$ .* We deduce by (2.9) that

$$0 \leq v_n \leq \mathbb{G}_{\mathcal{O}_n}[\nu_{n-1}] \quad \text{a.e. in } \mathbb{R}^N. \quad (3.5)$$

Observing that for any  $n \in \mathbb{N}$ ,

$$G_{\mathcal{O}_n}(x, y) \leq G_{\Omega}(x, y) \leq G_{\mathbb{R}^N}(x, y), \quad \text{for any } x, y \in \mathbb{R}^N, x \neq y$$

and  $\nu_{n-1} \leq \nu$ , we have that

$$\mathbb{G}_{\mathcal{O}_n}[\nu_{n-1}](x) \leq \int_{\Omega} G_{\Omega}(x, y) d\nu(y) = \mathbb{G}_{\Omega}[\nu](x) \leq \mathbb{G}_{\mathbb{R}^N}[\nu](x), \quad x \in \mathbb{R}^N,$$

where we make zero extension of  $\nu$  such that  $\nu \in \mathfrak{M}^b(\mathbb{R}^N)$ . Therefore, by (3.5) we obtain that for any  $n \in \mathbb{N}$ ,

$$v_n \leq \mathbb{G}_{\Omega}[\nu] \leq \mathbb{G}_{\mathbb{R}^N}[\nu] \quad \text{a.e. in } \mathbb{R}^N. \quad (3.6)$$

*Step 3. Existence of weak solution.* By (3.4) and (3.6), we see that the limit of  $\{v_n\}$  exists, denoted it by  $u_{\nu}$ . Hence,

$$0 \leq u_{\nu} \leq \mathbb{G}_{\Omega}[\nu] \leq \mathbb{G}_{\mathbb{R}^N}[\nu] \quad \text{a.e. in } \mathbb{R}^N. \quad (3.7)$$

It follows by Lemma 2.1 that  $\mathbb{G}_{\mathbb{R}^N}[\nu] \in L^1(\mathbb{R}^N, d\omega)$  and then  $u_{\nu} \in L^1(\mathbb{R}^N, d\omega)$ . Thus,  $v_n \rightarrow u_{\nu}$  in  $L^1(\mathbb{R}^N, d\omega)$  as  $n \rightarrow \infty$ . Moreover,

(i)  $\{h(v_n)\}_{n \in \mathbb{N}}$  is an increasing sequence of functions and  $h(v_n) \rightarrow h(u_{\nu})$  a.e. in  $\mathbb{R}^N$ ;

(ii) it implies by  $\Omega^c \subset \mathcal{O}_n^c$  and  $v_n = 0$  in  $\mathcal{O}_n^c$  that  $u_{\nu} = 0$  in  $\Omega^c$ .

For  $\xi \in \mathbb{X}_\Omega$ , there exists  $N_0 > 0$  such that for any  $n \geq N_0$ ,

$$\text{supp}(\xi) \subset \bar{\mathcal{O}}_n,$$

which implies that  $\xi \in \mathbb{X}_{\mathcal{O}_n}$  and then

$$\int_{\Omega} [v_n(-\Delta)^\alpha \xi + h(v_n)\xi] dx = \int_{\Omega} \xi d\nu_n. \quad (3.8)$$

By Lemma 3.1,

$$|(-\Delta)^\alpha \xi(x)| \leq \frac{c_9 \|\xi\|_{L^\infty(\Omega)}}{1 + |x|^{N+2\alpha}}, \quad x \in \Omega.$$

Thus,

$$\lim_{n \rightarrow \infty} \int_{\Omega} v_n(x) (-\Delta)^\alpha \xi(x) dx = \int_{\Omega} u_\nu(x) (-\Delta)^\alpha \xi(x) dx. \quad (3.9)$$

By (3.7) and increasing monotonicity of  $h$ , it follows  $h(u_\nu) \leq h(\mathbb{G}_{\mathbb{R}^N}[\nu])$  a.e. in  $\mathbb{R}^N$ . By Lemma 2.1, we find that  $h(u_\nu) \in L^1(\mathbb{R}^N, d\omega)$ . As a result, for any  $n \geq N_0$ ,

$$h(v_n) \rightarrow h(u_\nu) \quad \text{in} \quad L^1(\mathbb{R}^N, d\omega).$$

Consequently,

$$\lim_{n \rightarrow \infty} \int_{\Omega} h(v_n) \xi(x) dx = \int_{\Omega} h(u_\nu) \xi(x) dx. \quad (3.10)$$

It is obvious that

$$\lim_{n \rightarrow \infty} \int_{\Omega} \xi(x) d\nu_n = \int_{\Omega} \xi(x) d\nu. \quad (3.11)$$

Combining (3.9), (3.10) with (3.11) and taking  $n \rightarrow \infty$  in (3.8), we obtain that

$$\int_{\Omega} [u_\nu(-\Delta)^\alpha \xi + h(u_\nu)\xi] dx = \int_{\Omega} \xi d\nu. \quad (3.12)$$

Since  $\xi \in \mathbb{X}_\Omega$  is arbitrary,  $u_\nu$  is a weak solution of (1.1).  $\square$

## 4 Properties of weak solutions

In this section, we investigate problem (1.9). First we show that there is a unique solution of problem (1.9), then we establish the asymptotic behavior at the origin and infinity for the solution. In other words, we will prove Theorem 1.2 and Theorem 1.3.

#### 4.1 Properties of $u_k$

In this subsection, we consider the properties of nonnegative weak solution to (1.9). To this end, we introduce an auxiliary lemma.

**Lemma 4.1** [18, Lemma 3.1] *Assume that  $v \in C^{2\alpha+\epsilon}(\bar{B}_1)$  with  $\epsilon > 0$  satisfies*

$$(-\Delta)^\alpha v = \varphi \quad \text{in } B_1(0),$$

*where  $\varphi \in C^1(\bar{B}_1)$ . Then for  $\beta \in (0, 2\alpha)$ , there exists  $c_{11} > 0$  such that*

$$\|v\|_{C^\beta(\bar{B}_{1/4}(0))} \leq c_{11}(\|v\|_{L^\infty(B_1(0))} + \|\varphi\|_{L^\infty(B_1(0))} + \|v\|_{L^1(\mathbb{R}^N, d\omega)}).$$

**Lemma 4.2** *Assume that  $k > 0$ ,  $p \in (0, \frac{N}{N-2\alpha})$  and  $u$  is a nonnegative weak solution of (1.9). Then  $u$  is a classical solution of (1.11) and for any  $R > 1$ , there exists a weak solution  $u_R$  of*

$$\begin{aligned} (-\Delta)^\alpha u + u^p &= k\delta_0 \quad \text{in } B_R(0), \\ u &= 0 \quad \text{in } B_R^c(0) \end{aligned} \tag{4.1}$$

*such that*

$$u - m_R \leq u_R \leq u \quad \text{in } B_R(0) \setminus \{0\}, \tag{4.2}$$

*where  $m_R = \sup_{|x|>R} u(x)$ .*

**Proof.** The proof is divide into two parts. First we show the regularity of solution  $u$ , then we find  $u_R$  to establish the inequality (4.2).

*1. Regularity of  $u$ .* Let  $\{\eta_n\} \subset C_0^\infty(\mathbb{R}^N)$  be a sequence of radially decreasing and symmetric mollifiers such that  $\text{supp}(\eta_n) \subset B_{\varepsilon_n}(0)$  with  $\varepsilon_n \leq \frac{1}{n}$  and  $u_n = u * \eta_n$ . We observe that

$$u_n \rightarrow u \quad \text{and} \quad u_n^p \rightarrow u^p \quad \text{in } L^1(\mathbb{R}^N, d\omega) \quad \text{as } n \rightarrow \infty. \tag{4.3}$$

By Fourier transformation, we have that

$$\eta_n * (-\Delta)^\alpha \xi = (-\Delta)^\alpha (\xi * \eta_n),$$

then

$$\int_{\mathbb{R}^N} (u(-\Delta)^\alpha (\xi * \eta_n) + \xi * \eta_n u^p) dx = \int_{\mathbb{R}^N} (u * \eta_n (-\Delta)^\alpha \xi + \eta_n * u^p \xi) dx,$$

where  $\eta_n$  is radially symmetric. It follows that  $u_n$  is a classical solution of

$$\begin{aligned} (-\Delta)^\alpha u_n + u^p * \eta_n &= k\eta_n \quad \text{in } \mathbb{R}^N, \\ \lim_{|x| \rightarrow \infty} u_n(x) &= 0. \end{aligned} \tag{4.4}$$

We observe that  $0 \leq u_n \leq k\mathbb{G}_{\mathbb{R}^N}[\eta_n]$ , which implies  $0 \leq u \leq k\mathbb{G}_{\mathbb{R}^N}[\delta_0]$  in  $\mathbb{R}^N \setminus \{0\}$ . Since  $u^p \in L^1(\mathbb{R}^N, d\omega)$ , we have  $u^p * \eta_n \rightarrow u^p$  in  $L^1(\mathbb{R}^N, d\omega)$  as  $n \rightarrow \infty$  and that  $\{k\eta_n + u_n^p - u^p * \eta_n\}$  converges to  $k\delta_0$  in the distribution sense as  $n \rightarrow \infty$ .

By Lemma 2.2, we have that  $0 \leq u_n \leq \mathbb{G}_{\mathbb{R}^N}[k\eta_n]$  and  $\mathbb{G}_{\mathbb{R}^N}[k\eta_n]$  converges to  $\mathbb{G}_{\mathbb{R}^N}[k\delta_0]$  uniformly in any compact set of  $\mathbb{R}^N \setminus \{0\}$  and in  $L^1(\mathbb{R}^N, d\omega)$ . For a fixed  $r > 0$ , there exists  $N_0 > 0$  such that  $\text{supp}(\eta_n) \subset \bar{B}_r(0)$  and there exists  $c_{12} > 0$  such that for any  $n \geq N_0$ ,

$$\|u_n\|_{L^\infty(B_{r/2}^c(0))} \leq k\|\mathbb{G}_{\mathbb{R}^N}[\eta_n]\|_{L^\infty(B_{r/2}^c(0))} \leq c_{12}k\|\mathbb{G}_{\mathbb{R}^N}[\delta_0]\|_{L^\infty(B_{r/2}^c(0))}$$

and

$$\|u_n\|_{L^1(\mathbb{R}^N, d\omega)} \leq k\|\mathbb{G}_{\mathbb{R}^N}[\eta_n]\|_{L^1(\mathbb{R}^N, d\omega)} \leq c_{12}k\|\mathbb{G}_{\mathbb{R}^N}[\delta_0]\|_{L^1(\mathbb{R}^N, d\omega)}.$$

By Lemma 4.1, for any  $x_0 \in \mathbb{R}^N$  with  $|x_0| > 4r$ , there exists  $\beta \in (0, 2\alpha)$  such that

$$\begin{aligned} \|u_n\|_{C^\beta(B_{2r}(x_0))} &\leq c_{11}(\|u_n\|_{L^1(\mathbb{R}^N, d\omega)} + \|u^p * \eta_n\|_{L^\infty(B_{3r}(x_0))} + \|u_n\|_{L^\infty(B_{3r}(x_0))}) \\ &\leq c_{11}(c_{12}k\|\mathbb{G}_{\mathbb{R}^N}[\delta_0]\|_{L^1(\mathbb{R}^N, d\omega)} + \|u^p\|_{L^\infty(B_{3r}(x_0))} \\ &\quad + c_{12}k\|\mathbb{G}_{\mathbb{R}^N}[\delta_0]\|_{L^\infty(B_{r/2}^c(0))}) \\ &\leq c_{11}(c_{12}k\|\mathbb{G}_{\mathbb{R}^N}[\delta_0]\|_{L^1(\mathbb{R}^N, d\omega)} + k^p\|\mathbb{G}_{\mathbb{R}^N}[\delta_0]\|_{L^\infty(B_{r/2}^c(0))}^p \\ &\quad + c_{12}k\|\mathbb{G}_{\mathbb{R}^N}[\delta_0]\|_{L^\infty(B_{r/2}^c(0))}). \end{aligned}$$

Therefore, by the definition of  $u_n$  and Arzela-Ascoli Theorem, we obtain that  $u \in C^{\frac{\beta}{2}}(B_{2r}(x_0))$ . By Corollary 2.4 in [24], we deduce that

$$\begin{aligned} \|u_n\|_{C^{2\alpha+\frac{\beta}{2}}(B_r(x_0))} &\leq c_{13}(\|u_n\|_{L^1(\mathbb{R}^N, d\omega)} + \|u^p * \eta_n\|_{C^{\frac{\beta}{2}}(B_{2r}(x_0))} \\ &\quad + \|u_n\|_{C^{\frac{\beta}{2}}(B_{2r}(x_0))}) \\ &\leq c_{14}(k\|\mathbb{G}_{\mathbb{R}^N}[\delta_0]\|_{L^1(\mathbb{R}^N, d\omega)} + \|u\|_{C^{\frac{\beta}{2}}(B_{2r}(x_0))} \\ &\quad + k^p\|\mathbb{G}_{\mathbb{R}^N}[\delta_0]\|_{L^\infty(B_{r/2}^c(0))}^p + k\|\mathbb{G}_{\mathbb{R}^N}[\delta_0]\|_{L^\infty(B_{r/2}^c(0))}), \end{aligned}$$

where  $c_{13}, c_{14} > 0$ . Thus,  $u \in C^{2\alpha+\frac{\beta}{4}}(B_r(x_0))$  and by arbitrary of  $r > 0$  and  $x_0$ ,  $u$  is  $C^{2\alpha+\frac{\beta}{4}}$  locally in  $\mathbb{R}^N \setminus \{0\}$ . Therefore,  $u_n \rightarrow u$  and  $\eta_n \rightarrow 0$  uniformly in any compact subset of  $\mathbb{R}^N \setminus \{0\}$  as  $n \rightarrow \infty$ . We conclude that  $u$  is a classical solution of (1.11) by Corollary 4.6 in [10].

2. *Existence of  $u_R$ .* It infers from (4.4) that for given  $R > 1$ ,  $u_n$  is a classical solution of

$$\begin{aligned} (-\Delta)^\alpha u_n + u_n^p &= k\eta_n + u_n^p - u^p * \eta_n && \text{in } B_R(0), \\ u_n &\geq 0 && \text{in } B_R^c(0). \end{aligned}$$

We observe that for  $R > 1$ ,

$$\tilde{u}_n := (u - m_{R-\epsilon_n}) * \eta_n = u_n - m_{R-\epsilon_n} \leq u_n \quad \text{in } \mathbb{R}^N$$

and  $(-\Delta)^\alpha \tilde{u}_n = (-\Delta)^\alpha u_n$ , therefore,

$$(-\Delta)^\alpha \tilde{u}_n + |\tilde{u}_n|^{p-1} \tilde{u}_n = k\eta_n + |\tilde{u}_n|^{p-1} \tilde{u}_n - u^p * \eta_n \quad \text{in } B_R(0).$$

By the definition of  $m_{R-\epsilon_n}$ , we have  $u - m_{R-\epsilon_n} \leq 0$  in  $B_{R-\epsilon_n}^c(0)$ , and then

$$\tilde{u}_n \leq 0 \quad \text{in } B_R^c(0).$$

Let  $u_{n,R}$  be the solution of

$$\begin{aligned} (-\Delta)^\alpha u_{n,R} + u_{n,R}^p &= k\eta_n + u_n^p - u^p * \eta_n && \text{in } B_R(0), \\ u_{n,R} &= 0 && \text{in } B_R^c(0). \end{aligned}$$

By Lemma 2.2, we have that

$$\tilde{u}_n \leq u_{n,R} \leq u_n \quad \text{in } \mathbb{R}^N. \quad (4.5)$$

It is known that  $u_R := \lim_{n \rightarrow \infty} u_{n,R}$  is a weak solution of (4.1), since  $\{k\eta_n + u_n^p - u^p * \eta_n\}$  converges to  $k\delta_0$  in the distribution sense as  $n \rightarrow \infty$ . Hence, (4.2) follows by (4.5) and  $\tilde{u}_n \rightarrow u - m_R$  in  $\mathbb{R}^N$  as  $n \rightarrow \infty$ .  $\square$

With the help of Lemma 4.2, we show next the uniqueness of weak solution to (1.9).

**Proposition 4.1** *Assume that  $k > 0$  and  $0 < p < \frac{N}{N-2\alpha}$ . Then (1.9) admits a unique weak solution  $u_k$ .*

**Proof.** *Existence.* By Theorem 1.1, there exists at least one weak solution  $u_k$  to

$$(-\Delta)^\alpha u + u^p = k\delta_0 \quad \text{in } \mathbb{R}^N$$

such that  $0 \leq u_k \leq k\mathbb{G}_{\mathbb{R}^N}[\delta_0]$  a.e. in  $\mathbb{R}^N$ . We observe that

$$\mathbb{G}_{\mathbb{R}^N}[\delta_0](x) = \frac{c_{N,\alpha}}{|x|^{N-2\alpha}}, \quad x \in \mathbb{R}^N \setminus \{0\},$$

then

$$\lim_{|x| \rightarrow \infty} u_k(x) = 0.$$

Thus  $u_k$  is a weak solution of (1.9).

*Uniqueness.* We assume that  $u_k, \tilde{u}_k$  are two different weak solutions of (1.9) and

$$A_0 := \min\{1, \limsup_{x \rightarrow 0} |\tilde{u}_k - u_k|(x)\}.$$



We claim that  $A_0 > 0$ . In fact, if not, then  $\lim_{x \rightarrow 0} |\tilde{u}_k - u_k|(x) = 0$ . Now we may assume that there exists  $x_0 \in \mathbb{R}^N \setminus \{0\}$  such that

$$(\tilde{u}_k - u_k)(x_0) = \sup_{x \in \mathbb{R}^N \setminus \{0\}} (\tilde{u}_k - u_k)(x) > 0,$$

which implies that

$$(-\Delta)^\alpha(\tilde{u}_k - u_k)(x_0) \geq 0.$$

Then we obtain a contradiction by the fact that  $\tilde{u}_k$  and  $u_k$  are classical solutions of (1.11) by Lemma 4.2. Therefore,  $A_0 > 0$ . Since

$$\lim_{|x| \rightarrow \infty} u_k(x) = 0 \quad \text{and} \quad \lim_{|x| \rightarrow \infty} \tilde{u}_k(x) = 0,$$

for  $R > 0$  large enough,

$$\epsilon_1 := \sup_{|x| \geq R} u_k(x) \leq \frac{A_0}{2} \quad \text{and} \quad \epsilon_2 := \sup_{|x| \geq R} \tilde{u}_k(x) \leq \frac{A_0}{2}.$$

Since  $u_k$  and  $\tilde{u}_k$  are weak solutions of (1.9), by Lemma 4.2, there exist weak solutions  $u_{k,R}$  and  $\tilde{u}_{k,R}$  to (4.1) such that

$$u_k - \epsilon_1 \leq u_{k,R} \leq u_k \quad \text{in} \quad B_R(0) \setminus \{0\} \quad (4.6)$$

and

$$\tilde{u}_k - \epsilon_2 \leq \tilde{u}_{k,R} \leq \tilde{u}_k \quad \text{in} \quad B_R(0) \setminus \{0\}. \quad (4.7)$$

Moreover, by Proposition 2.3 we obtain

$$u_{k,R} \equiv \tilde{u}_{k,R},$$

which, together with (4.7) and (4.6), implies that

$$|u_k - \tilde{u}_k| \leq \max\{\epsilon_1, \epsilon_2\} \quad \text{in} \quad B_R(0) \setminus \{0\}.$$

Thus,

$$\limsup_{x \rightarrow 0} |u_k - \tilde{u}_k|(x) \leq \max\{\epsilon_1, \epsilon_2\} < A_0.$$

This contradicts to the definition of  $A_0$ . As a consequence, problem (1.9) has a unique weak solution.  $\square$

Now we estimate the singularity rate of weak solution to (1.9) at the origin.

**Proposition 4.2** *Let  $k > 0$ ,  $0 < p < \frac{N}{N-2\alpha}$  and  $u_k$  be the weak solution of (1.9). Then*

$$\lim_{x \rightarrow 0} u_k(x) |x|^{N-2\alpha} = c_{N,\alpha} k. \quad (4.8)$$

**Proof.** On the one hand, we have that

$$u_k(x) \leq \mathbb{G}_{\mathbb{R}^N}[k\delta_0](x) = c_{N,\alpha}k|x|^{-N+2\alpha}, \quad x \in \mathbb{R}^N \setminus \{0\}. \quad (4.9)$$

On the other hand, from the proof of Theorem 1.1 and uniqueness of weak solution to (1.9), we know  $u_k = \lim_{R \rightarrow \infty} u_{k,R}$ , where  $u_{k,R}$  is the weak solution of (4.1). By [18, Lemma 2.1] and [18, Proposition 1.1], we have that

$$\lim_{x \rightarrow 0} u_{k,R}(x)|x|^{N-2\alpha} = c_{N,\alpha}k.$$

Then together with (4.9) and the fact that  $\{u_{k,R}\}_R$  is an increasing sequence of functions, (4.8) holds.  $\square$

**Proof of Theorem 1.2.** By Proposition 4.1, Lemma 4.2 and (4.8), the assertion of Theorem 1.2 holds except part (ii).

Now, we prove part (ii) of Theorem 1.2. In fact, let  $k_1 \leq k_2$  and  $u_{k_1}, u_{k_2}$  be the solution of (1.9) with  $k = k_1$  and  $k = k_2$ , respectively. For  $R > 1$ , we denote by  $u_{k_1,R}$  and  $u_{k_2,R}$  the solutions of (4.1) with  $k = k_1$  and  $k = k_2$ , respectively. By  $k_1 \leq k_2$  and Proposition 2.3, we have that

$$u_{k_1,R} \leq u_{k_2,R} \quad \text{in } \mathbb{R}^N \setminus \{0\}.$$

Similar to the proof of Theorem 1.1, we know that  $u_{k_i} = \lim_{R \rightarrow \infty} u_{k_i,R}$  with  $i = 1, 2$ . Therefore,  $u_{k_1} \leq u_{k_2}$  in  $\mathbb{R}^N \setminus \{0\}$ .  $\square$

## 4.2 Asymptotic behavior of $u_1$ at $\infty$

This subsection is devoted to investigate the asymptotic behavior of weak solution  $u_1$  at  $\infty$  to

$$\begin{aligned} (-\Delta)^\alpha u + u^p &= \delta_0 \quad \text{in } \mathbb{R}^N, \\ \lim_{|x| \rightarrow +\infty} u(x) &= 0, \end{aligned} \quad (4.10)$$

where  $p \in (1, \frac{N}{N-2\alpha})$ . We observe that

$$\lim_{x \rightarrow 0} u_1(x)|x|^{N-2\alpha} = c_{N,\alpha}$$

and  $u_1$  is a classical solution of

$$(-\Delta)^\alpha u + u^p = 0 \quad \text{in } \mathbb{R}^N \setminus \{0\}. \quad (4.11)$$

In order to prove Theorem 1.3, we introduce some auxiliary lemmas. For  $\tau \in (-\infty, -N + 2\alpha)$ , we denote by  $w_\tau$  a  $C^2$  nonnegative radially symmetric function in  $\mathbb{R}^N$  such that  $w_\tau$  is decreasing in  $|x|$  and for  $|x| > 1$ ,

$$w_\tau(x) = \begin{cases} |x|^\tau & \text{for } \tau \in (-\infty, -N + 2\alpha) \setminus \{-N\}, \\ |x|^\tau \log^{\gamma_0}(\varrho_0 + |x|) & \text{for } \tau = -N, \end{cases} \quad (4.12)$$

where  $\varrho_0 = e^{\frac{1}{2\alpha}}$  and  $\gamma_0 = \frac{N}{2\alpha}$ .

**Lemma 4.3** Assume that  $\tau \in (-\infty, -N + 2\alpha)$ . Then

(i) for  $\tau \in (-\infty, -N)$ , there exist  $R \geq 4$  and  $c_{15} > 1$  such that for  $|x| > R$ ,

$$\frac{1}{c_{15}}|x|^{-N-2\alpha} \leq -(-\Delta)^\alpha w_\tau(x) \leq c_{15}|x|^{-N-2\alpha}; \quad (4.13)$$

(ii) for  $\tau = -N$ , there exist  $R \geq 4$  and  $c_{15} > 1$  such that for  $|x| > R$ ,

$$\frac{1}{c_{15}}|x|^{-N-2\alpha} \log^{\gamma_0+1} |x| \leq -(-\Delta)^\alpha w_\tau(x) \leq c_{15}|x|^{-N-2\alpha} \log^{\gamma_0+1} |x|; \quad (4.14)$$

(iii) for  $\tau \in (-N, -N + 2\alpha)$ , there exist  $R \geq 4$  and  $c_{15} > 1$  such that for  $|x| > R$ ,

$$\frac{1}{c_{15}}|x|^{\tau-2\alpha} \leq -(-\Delta)^\alpha w_\tau(x) \leq c_{15}|x|^{\tau-2\alpha}. \quad (4.15)$$

**Proof.** In the following, we shall use the equivalent definition of  $(-\Delta)^\alpha w_\tau$ , that is,

$$(-\Delta)^\alpha w_\tau(x) = -\frac{1}{2} \int_{\mathbb{R}^N} \frac{w_\tau(x+z) + w_\tau(x-z) - 2w_\tau(x)}{|z|^{N+2\alpha}} dz.$$

(i) The case of  $\tau \in (-\infty, -N)$ . On the one hand, for  $|x| > 4$ , we have that

$$\begin{aligned} -(-\Delta)^\alpha w_\tau(x) &= \frac{1}{2} \int_{\mathbb{R}^N \setminus (B_1(x) \cup B_1(-x))} \frac{w_\tau(x+z) + w_\tau(x-z) - 2w_\tau(x)}{|z|^{N+2\alpha}} dz \\ &\quad + \frac{1}{2} \int_{B_1(x) \cup B_1(-x)} \frac{w_\tau(x+z) + w_\tau(x-z) - 2w_\tau(x)}{|z|^{N+2\alpha}} dz \\ &\leq \frac{|x|^{\tau-2\alpha}}{2} \int_{D_0} \frac{I_x(y)}{|y|^{N+2\alpha}} dy + c_{16}|x|^{-N-2\alpha} \end{aligned} \quad (4.16)$$

where  $c_{16} > 0$  depends on  $\|w\|_{L^1(B_1(0))}$ ,  $e_x = \frac{x}{|x|}$ ,  $D_0 = \mathbb{R}^N \setminus (B_{\frac{1}{|x|}}(e_x) \cup B_{\frac{1}{|x|}}(-e_x))$  and

$$I_x(y) = |e_x + y|^\tau + |e_x - y|^\tau - 2.$$

On the other hand, for  $|x| \geq 4$ ,

$$\begin{aligned} -(-\Delta)^\alpha w_\tau(x) &= \frac{1}{2} \int_{\mathbb{R}^N \setminus (B_1(x) \cup B_1(-x))} \frac{w_\tau(x+z) + w_\tau(x-z) - 2w_\tau(x)}{|z|^{N+2\alpha}} dz \\ &\quad + \frac{1}{2} \int_{B_1(x) \cup B_1(-x)} \frac{w_\tau(x+z) + w_\tau(x-z) - 2w_\tau(x)}{|z|^{N+2\alpha}} dz \\ &\geq \frac{|x|^{\tau-2\alpha}}{2} \int_{D_0} \frac{I_x(y)}{|y|^{N+2\alpha}} dy - 2 \int_{B_1(x)} \frac{w_\tau(x)}{|z|^{N+2\alpha}} dz \\ &\geq \frac{|x|^{\tau-2\alpha}}{2} \int_{D_0} \frac{I_x(y)}{|y|^{N+2\alpha}} dy - c_{17}|x|^{\tau-N-2\alpha}, \end{aligned} \quad (4.17)$$

where  $c_{17} > 0$ .

*Claim 1.* There exists  $c_{18} > 1$  such that

$$\frac{1}{c_{18}}|x|^{-N-\tau} \leq \int_{D_1 \cup D_2} \frac{I_x(y)}{|y|^{N+2\alpha}} dy \leq c_{18}|x|^{-N-\tau}, \quad (4.18)$$

where  $D_1 = B_{\frac{1}{2}}(-e_x) \setminus B_{\frac{1}{|x|}}(-e_x)$  and  $D_2 = B_{\frac{1}{2}}(e_x) \setminus B_{\frac{1}{|x|}}(e_x)$ .

In fact, for  $y \in D_1$ , we observe that

$$-2 \leq |e_x - y|^\tau - 2 \leq -1 \quad \text{and} \quad \frac{1}{2} \leq |y| \leq \frac{3}{2},$$

then

$$\begin{aligned} \int_{D_1} \frac{I_x(y)}{|y|^{N+2\alpha}} dy &\leq c_{19} \int_{B_{\frac{1}{2}}(0) \setminus B_{\frac{1}{|x|}}(0)} |y|^\tau dy + \int_{D_1} \frac{|e_x - y|^\tau - 2}{|y|^{N+2\alpha}} dy \\ &\leq c_{20} \int_{|x|^{-1}}^{\frac{1}{2}} r^{\tau+N-1} dr \\ &\leq c_{21} |x|^{-N-\tau} \end{aligned}$$

and

$$\begin{aligned} \int_{D_1} \frac{I_x(y)}{|y|^{N+2\alpha}} dy &\geq c_{22} \int_{|x|^{-1}}^{\frac{1}{2}} r^{\tau+N-1} dr - c_{23} \\ &\geq c_{24} |x|^{-N-\tau} - c_{23}, \end{aligned}$$

where  $c_{18}, \dots, c_{24}$  are positive constants. Since  $-N-\tau > 0$ , there exist  $R \geq 4$  and  $c_{25} > 0$  such that for  $|x| \geq R$ ,

$$\frac{1}{c_{25}} |x|^{-N-\tau} \leq \int_{D_1} \frac{I_x(y)}{|y|^{N+2\alpha}} dy \leq c_{25} |x|^{-N-\tau}.$$

By the fact

$$\int_{D_1} \frac{I_x(y)}{|y|^{N+2\alpha}} dy = \int_{D_2} \frac{I_x(y)}{|y|^{N+2\alpha}} dy,$$

we obtain (4.18).

*Claim 2. There exists  $c_{26} > 0$  such that*

$$\left| \int_{B_{\frac{1}{2}}(0)} \frac{I_x(y)}{|y|^{N+2\alpha}} dy \right| \leq c_{26}. \quad (4.19)$$

Indeed, since function  $I_x$  is  $C^2$  in  $\bar{B}_{\frac{1}{2}}(0)$  such that

$$I_x(0) = 0 \quad \text{and} \quad I_x(y) = I_x(-y), \quad y \in \bar{B}_{\frac{1}{2}}(0),$$

then  $\nabla I_x(0) = 0$  and there exists  $c_{27} > 0$  such that

$$|D^2 I_x(y)| \leq c_{27}, \quad y \in B_{\frac{1}{2}}(0).$$

Then we have

$$I_x(y) \leq c_{27} |y|^2, \quad y \in B_{\frac{1}{2}}(0),$$

which implies that

$$|\int_{B_{\frac{1}{2}}(0)} \frac{I_x(y)}{|y|^{N+2\alpha}} dy| \leq c_{27} \int_{B_{\frac{1}{2}}(0)} \frac{|y|^2}{|y|^{N+2\alpha}} dy \leq c_{26}.$$

*Claim 3. There exists  $c_{28} > 0$  such that*

$$|\int_{D_3} \frac{I_x(y)}{|y|^{N+2\alpha}} dy| \leq c_{28}, \quad (4.20)$$

where  $D_3 = \mathbb{R}^N \setminus (B_{\frac{1}{2}}(0) \cup B_{\frac{1}{2}}(e_x) \cup B_{\frac{1}{2}}(-e_x)) = D_0 \setminus (D_1 \cup D_2 \cup B_{\frac{1}{2}}(0))$ .

In fact, for  $y \in D_3$ , we observe that there exists  $c_{29} > 0$  such that  $|I_x(y)| \leq c_{29}$  and

$$|\int_{D_3} \frac{I_x(y)}{|y|^{N+2\alpha}} dy| \leq \int_{\mathbb{R}^N \setminus B_{\frac{1}{2}}(0)} \frac{c_{29}}{|y|^{N+2\alpha}} dy \leq c_{30},$$

where  $c_{30} > 0$ . Since  $\lim_{|x| \rightarrow \infty} |x|^{-N-\tau} = \infty$  for  $\tau < -N$ , by (4.18)-(4.20), there exist  $R \geq 4$  and  $c_{31} > 1$  such that for  $|x| \geq R$ ,

$$\frac{1}{c_{31}} |x|^{-N-2\alpha} \leq -(-\Delta)^\alpha w(x) \leq c_{31} |x|^{-N-2\alpha}. \quad (4.21)$$

(ii) The case of  $\tau = -N$ . Similarly to (4.16) and (4.17), we have that for  $|x| > 4$ ,

$$-(-\Delta)^\alpha w_\tau(x) \leq \frac{|x|^{-N-2\alpha} \log^{\gamma_0}(\varrho_0 + |x|)}{2} \int_{D_0} \frac{II_x(y)}{|y|^{N+2\alpha}} dy + c_{32} |x|^{-N-2\alpha}$$

and

$$-(-\Delta)^\alpha w_\tau(x) \geq \frac{|x|^{-N-2\alpha} \log^{\gamma_0}(\varrho_0 + |x|)}{2} \int_{D_0} \frac{II_x(y)}{|y|^{N+2\alpha}} dy - c_{32} |x|^{-N-2\alpha} \log^{\gamma_0} |x|,$$

where  $c_{32} > 0$  and

$$II_x(y) = \frac{\log^{\gamma_0}(\varrho_0 + |x| |e_x + y|)}{\log^{\gamma_0}(\varrho_0 + |x|)} |e_x + y|^{-N} + \frac{\log^{\gamma_0}(\varrho_0 + |x| |e_x - y|)}{\log^{\gamma_0}(\varrho_0 + |x|)} |e_x - y|^{-N} - 2.$$

For  $y \in D_1$ , we have that  $\frac{\log^{\gamma_0}(\varrho_0 + |x| |e_x + y|)}{\log^{\gamma_0}(\varrho_0 + |x|)} \leq 1$ , then

$$\begin{aligned} \int_{D_1} \frac{II_x(y)}{|y|^{N+2\alpha}} dy &\leq c_{33} \int_{B_{\frac{1}{2}}(0) \setminus B_{\frac{1}{|x|}}(0)} |y|^{-N} dy + c_{34} \\ &\leq c_{35} \log |x| + c_{34} \end{aligned}$$

and

$$\begin{aligned} \int_{D_1} \frac{II_x(y)}{|y|^{N+2\alpha}} dy &\geq c_{33} \int_{|x|^{-1}}^{\frac{1}{2}} r^{-1} \frac{\log^{\gamma_0}(\varrho_0 + |x|r)}{\log^{\gamma_0}(\varrho_0 + |x|)} dr - c_{34} \\ &= c_{33} \int_1^{\frac{1}{2}|x|} s^{-1} \frac{\log^{\gamma_0}(\varrho_0 + s)}{\log^{\gamma_0}(\varrho_0 + |x|)} ds - c_{34} \\ &\geq c_{35} \log |x| - c_{34}, \end{aligned}$$

where  $c_{33}, c_{34}, c_{35} > 0$ . By the fact that

$$\int_{D_1} \frac{II_x(y)}{|y|^{N+2\alpha}} dy = \int_{D_2} \frac{II_x(y)}{|y|^{N+2\alpha}} dy,$$

there exists  $c_{36} > 0$  such that

$$\frac{1}{c_{36}} \log |x| \leq \int_{D_1 \cup D_2} \frac{II_x(y)}{|y|^{N+2\alpha}} dy \leq c_{36} \log |x|. \quad (4.22)$$

Similarly to (4.18)-(4.20), there exists  $c_{37} > 0$  such that

$$\left| \int_{\mathbb{R}^N \setminus (B_{\frac{1}{2}}(e_x) \cup B_{\frac{1}{2}}(-e_x))} \frac{II_x(y)}{|y|^{N+2\alpha}} dy \right| \leq c_{37}, \quad (4.23)$$

which, together with (4.22), imply (4.14).

(iii) The case that  $\tau \in (-N, -N + 2\alpha)$ . By Lemma 3.1 and Lemma 3.2 in [19], we have

$$(-\Delta)^\alpha |x|^\tau = c(\tau) |x|^{\tau-2\alpha}, \quad (4.24)$$

where  $c(\tau) < 0$ .

Let  $\tilde{w}(x) = w_\tau(x) - |x|^\tau$  for  $x \in \mathbb{R}^N \setminus \{0\}$ , then  $\tilde{w} = 0$  in  $B_1^c(0)$ . For  $|x| > 4$ , we have that

$$\begin{aligned} |(-\Delta)^\alpha \tilde{w}(x)| &\leq \int_{B_1(0)} \frac{w_\tau(z) + |z|^\tau}{|z-x|^{N+2\alpha}} dz \\ &\leq (|x| - 1)^{-N-2\alpha} \int_{B_1(0)} (w_\tau(z) + |z|^\tau) dz, \end{aligned} \quad (4.25)$$

which, together with (4.24), imply (4.15). We complete the proof.  $\square$

**Lemma 4.4** *Let  $\eta : \mathbb{R}^N \rightarrow [0, 1]$  be a  $C^2$  function with support in  $B_2(0)$  and  $\eta = 1$  in  $B_1(0)$ ,  $\bar{w}(x) = \eta(x)|x|^{-N+2\alpha}$  for  $x \in \mathbb{R}^N$ . Then for  $|x| > 4$ , there exists  $c_{38} > 1$  such that*

$$\frac{1}{c_{38}} |x|^{-N-2\alpha} \leq -(-\Delta)^\alpha \bar{w}(x) \leq c_{38} |x|^{-N-2\alpha}, \quad x \in B_4^c(0). \quad (4.26)$$

**Proof.** For  $|x| > 4$ , we have that

$$-(-\Delta)^\alpha \bar{w}(x) = \int_{B_2(0)} \frac{\bar{w}(z)}{|z-x|^{N+2\alpha}} dz \leq (|x| - 2)^{-N-2\alpha} \int_{B_2(0)} \bar{w}(z) dz$$

and

$$-(-\Delta)^\alpha \bar{w}(x) \geq (|x| + 2)^{-N-2\alpha} \int_{B_2(0)} \bar{w}(z) dz,$$

which, together with  $\int_{B_2(0)} \bar{w}(z) dz < +\infty$ , imply (4.26).  $\square$

Now we are in the position to prove Theorem 1.3.

**Proof of Theorem 1.3.** For  $p \in (1, \frac{N}{N-2\alpha})$ , we denote

$$\tau_p = \begin{cases} -\frac{2\alpha}{p-1} & \text{for } p \in [1 + \frac{2\alpha}{N}, \frac{N}{N-2\alpha}), \\ -\frac{N+2\alpha}{p} & \text{for } p \in (1, 1 + \frac{2\alpha}{N}). \end{cases} \quad (4.27)$$

We note that  $\tau_p$  is continuous and strictly increasing with respect to  $p$ ,  $\tau_p = -N$  if  $p = 1 + \frac{2\alpha}{N}$  and  $\lim_{p \rightarrow 0^+} \tau_p = -\infty$ .

*Lower bound.* Since  $\lim_{|x| \rightarrow 0} u_1(x) = \infty$  and  $u_1$  is continuous and positive in  $\mathbb{R}^N \setminus \{0\}$ , then there exists  $c_{39} > 0$  such that

$$c_{39}w_{\tau_p} \leq u_1 \quad \text{in } \bar{B}_R(0) \setminus \{0\}, \quad (4.28)$$

where  $R > 4$  is from Lemma 4.3.

We note that for  $p \in (1, 1 + \frac{2\alpha}{N})$ ,  $\tau_p p = -N - 2\alpha$ ; for  $p = 1 + \frac{2\alpha}{N}$ ,  $\tau_p = -N$ ,  $\gamma_0 + 1 = p\gamma_0$  and for  $p \in (1 + \frac{2\alpha}{N}, \frac{N}{N-2\alpha})$ ,  $\tau_p - 2\alpha = p\tau_p$ . By Lemma 4.3, there exists  $t_0 \in (0, 1)$  such that

$$(-\Delta)^\alpha(t_0 c_{39} w_{\tau_p}) + (t_0 c_{39} w_{\tau_p})^p \leq 0 \quad \text{in } B_R^c(0).$$

We claim that  $u_1 \geq t_0 c_{39} w_{\tau_p}$  in  $B_R^c(0)$ . In fact, if not, there would exist  $x_0 \in B_R^c(0)$  such that

$$\begin{aligned} (u_1 - t_0 c_{39} w_{\tau_p})(x_0) &= \min_{x \in B_R^c(0)} (u_1 - t_0 c_{39} w_{\tau_p})(x) \\ &= \min_{x \in \mathbb{R}^N \setminus \{0\}} (u_1 - t_0 c_{39} w_{\tau_p})(x) < 0, \end{aligned}$$

since  $u_1 - t_0 c_{39} w_{\tau_p} \geq 0$  in  $\bar{B}_R(0)$  and  $\lim_{|x| \rightarrow \infty} (u_1 - t_0 c_{39} w_{\tau_p})(x) = 0$ . Then  $(-\Delta)^\alpha(u_1 - t_0 c_{39} w_{\tau_p})(x_0) < 0$ . However,

$$(-\Delta)^\alpha(u_1 - t_0 c_{39} w_{\tau_p})(x_0) \geq -u_1^p(x_0) + (t_0 c_{39} w_{\tau_p})^p(x_0) > 0,$$

which is a contradiction.

*Upper bound.* Since  $\lim_{x \rightarrow 0} u_1(x)|x|^{N-2\alpha} = c_{N,\alpha}$ , there exists  $c_{40} > 0$  such that  $u_1(x) \leq c_{40}|x|^{-N+2\alpha}$  in  $B_1(0) \setminus \{0\}$ . Then there exists  $c_{41} > 1$  such that

$$u_1 \leq c_{40}w_{\tau_p} + c_{41}\bar{w} \quad \text{in } \bar{B}_R(0),$$

where  $\bar{w}$  is from Lemma 4.4. Denote by  $W = c_{40}w_{\tau_p} + c_{41}\bar{w}$ . By Lemma 4.3 and Lemma 4.4, there exist  $t_1 > 1$  such that

$$(-\Delta)^\alpha(t_1 W) + (t_1 W)^p \geq 0 \quad \text{in } B_R^c(0).$$

We claim that  $u_1 \leq t_1 W$  in  $B_R^c(0)$ . In fact, if not, then there exists  $x_1 \in B_R^c(0)$  such that

$$\begin{aligned} (u_1 - t_1 W)(x_1) &= \max_{x \in B_R^c(0)} (u_1 - t_1 W)(x) \\ &= \max_{x \in \mathbb{R}^N \setminus \{0\}} (u_1 - t_1 W)(x) > 0. \end{aligned}$$

Thus,  $(-\Delta)^\alpha(u_1 - t_1 W)(x_1) > 0$ . But

$$(-\Delta)^\alpha(u_1 - t_1 W)(x_1) \leq -u_1^p(x_1) + (t_1 W)^p(x_1) < 0,$$

we obtain a contradiction.

Since  $\bar{w} = 0$  in  $B_2^c$ , combining (4.12) with (4.27), we obtain the decays of  $u_1$  for  $p \in (1, \frac{N}{N-2\alpha})$ .  $\square$

## 5 Properties of the limit function

Let  $u_\infty$  be given by (1.15) and  $u_{k,\Omega}$  be a weak solution of (1.16) when  $\Omega$  is an unbounded regular domain including the origin. We plan to study properties of both  $u_\infty$  and  $u_{k,\Omega}$ .

### 5.1 Properties of $u_\infty$

This subsection is devoted to prove Theorem 1.4. To this end, we introduce some propositions.

**Proposition 5.1** *Assume that  $p \in (1, \frac{N}{N-2\alpha})$  and  $u_\infty$  is defined in (1.15). Then*

$$u_\infty(x) = |x|^{-\frac{2\alpha}{p-1}} u_\infty\left(\frac{x}{|x|}\right), \quad x \in \mathbb{R}^N \setminus \{0\}. \quad (5.1)$$

**Proof.** For  $\lambda > 0$ , we denote

$$\tilde{u}_\lambda(x) = \lambda^{\frac{2\alpha}{p-1}} u_k(\lambda x), \quad x \in \mathbb{R}^N \setminus \{0\},$$

where  $u_k$  is the solution of (1.9). By direct computation, we have for  $x \in \mathbb{R}^N \setminus \{0\}$  that,

$$\begin{aligned} (-\Delta)^\alpha \tilde{u}_\lambda(x) + \tilde{u}_\lambda^p(x) &= \lambda^{\frac{2\alpha p}{p-1}} [(-\Delta)^\alpha u_k(\lambda x) + u_k^p(\lambda x)] \\ &= 0. \end{aligned} \quad (5.2)$$

Moreover, for  $f \in C_0(\mathbb{R}^N)$ ,

$$\begin{aligned} \langle (-\Delta)^\alpha \tilde{u}_\lambda + \tilde{u}_\lambda^p, f \rangle &= \lambda^{\frac{2\alpha p}{p-1}} \int_{\mathbb{R}^N} [(-\Delta)^\alpha u_k(\lambda x) + u_k^p(\lambda x)] f(x) dx \\ &= \lambda^{\frac{2\alpha p}{p-1} - N} \int_{\mathbb{R}^N} [(-\Delta)^\alpha u_k(z) + u_k^p(z)] f\left(\frac{z}{\lambda}\right) dz \\ &= \lambda^{\frac{2\alpha p}{p-1} - N} k f(0), \end{aligned}$$

where  $\frac{2\alpha p}{p-1} - N > 0$  by the fact that  $p \in (1, \frac{N}{N-2\alpha})$ . Thus,

$$(-\Delta)^\alpha \tilde{u}_\lambda + \tilde{u}_\lambda^p = \lambda^{\frac{2\alpha p}{p-1} - N} k \delta_0 \quad \text{in } \mathbb{R}^N. \quad (5.3)$$

We observe that  $\lim_{|x| \rightarrow \infty} \tilde{u}_\lambda(x) = 0$  and  $u_{k\lambda^{\frac{2\alpha p}{p-1} - N}}$  is a unique weak solution of (1.9) with  $k$  replaced by  $\lambda^{\frac{2\alpha p}{p-1} - N} k$ , then for  $x \in \mathbb{R}^N \setminus \{0\}$ ,

$$u_{k\lambda^{\frac{2\alpha p}{p-1} - N}}(x) = \tilde{u}_\lambda(x) = \lambda^{\frac{2\alpha}{p-1}} u_k(\lambda x) \quad (5.4)$$

and letting  $k \rightarrow \infty$  we have that

$$u_\infty(x) = \lambda^{\frac{2\alpha}{p-1}} u_\infty(\lambda x), \quad x \in \mathbb{R}^N \setminus \{0\},$$

which implies (5.1) by taking  $\lambda = |x|^{-1}$ .  $\square$



**Proposition 5.2** Suppose that  $p \in (0, 1 + \frac{2\alpha}{N}]$  and  $u_\infty$  is given by (1.15). Then

$$u_\infty = \infty \quad \text{in } \mathbb{R}^N.$$

**Proof.** In the case of  $p \in (0, 1]$ . We observe that

$$\mathbb{G}_{\mathbb{R}^N}[\delta_0], \quad \mathbb{G}_{\mathbb{R}^N}[(\mathbb{G}_{\mathbb{R}^N}[\delta_0])^p] > 0$$

in  $\mathbb{R}^N$ . Since

$$u_k \geq k\mathbb{G}_{\mathbb{R}^N}[\delta_0] - k^p\mathbb{G}_{\mathbb{R}^N}[(\mathbb{G}_{\mathbb{R}^N}[\delta_0])^p],$$

we obtain  $\lim_{k \rightarrow \infty} u_k = \infty$  in  $\mathbb{R}^N$  for  $p \in (0, 1)$ . For  $p = 1$ , we see that  $u_k = ku_1$ . Hence,  $\lim_{k \rightarrow \infty} u_k = \infty$  in  $\mathbb{R}^N$  by the fact that  $u_1 > 0$  in  $\mathbb{R}^N$ .

In the case of  $p \in (1, 1 + \frac{2\alpha}{N}]$ . It derives from (5.1) that

$$u_\infty(x) \geq c_{42}|x|^{-\frac{2\alpha}{p-1}}, \quad x \in \mathbb{R}^N \setminus \{0\},$$

where  $c_{42} = \min_{|x|=1} u_\infty(x) > 0$ , since  $u_\infty \geq u_k$  in  $\mathbb{R}^N \setminus \{0\}$ . Since  $u_\infty = \lim_{k \rightarrow \infty} u_k$  in  $\mathbb{R}^N \setminus \{0\}$ , we deduce that

$$\pi_k := \int_{B_{\frac{1}{4}}(0)} u_k(x) dx \rightarrow \infty \quad \text{as } k \rightarrow \infty. \quad (5.5)$$

Fix  $y_0 \in \mathbb{R}^N$  such that  $|y_0| = 1$ , it follows by Lemma 2.4 that problem

$$\begin{aligned} (-\Delta)^\alpha u + u^p &= 0 & \text{in } B_{\frac{1}{4}}(y_0), \\ u &= 0 & \text{in } \mathbb{R}^N \setminus (B_{\frac{1}{4}}(y_0) \cup B_{\frac{1}{4}}(0)), \\ u &= u_k & \text{in } B_{\frac{1}{4}}(0) \end{aligned} \quad (5.6)$$

admits a unique solution  $w_k$ . By Lemma 2.2,

$$u_k \geq w_k \quad \text{in } B_{\frac{1}{4}}(y_0). \quad (5.7)$$

Let  $\tilde{w}_k = w_k - u_k \chi_{B_{\frac{1}{4}}(0)}$ , then  $\tilde{w}_k = w_k$  in  $B_{\frac{1}{4}}(y_0)$  and for  $x \in B_{\frac{1}{4}}(y_0)$ ,

$$\begin{aligned} (-\Delta)^\alpha \tilde{w}_k(x) &= -\lim_{\epsilon \rightarrow 0^+} \int_{B_{\frac{1}{4}}(y_0) \setminus B_\epsilon(x)} \frac{w_k(z) - w_k(x)}{|z-x|^{N+2\alpha}} dz \\ &\quad + \lim_{\epsilon \rightarrow 0^+} \int_{B_{\frac{1}{4}}^c(y_0) \setminus B_\epsilon(x)} \frac{w_k(x)}{|z-x|^{N+2\alpha}} dz \\ &= -\lim_{\epsilon \rightarrow 0^+} \int_{\mathbb{R}^N \setminus B_\epsilon(x)} \frac{w_k(z) - w_k(x)}{|z-x|^{N+2\alpha}} dz + \int_{B_{\frac{1}{4}}(0)} \frac{u_k(z)}{|z-x|^{N+2\alpha}} dz \\ &\geq (-\Delta)^\alpha w_k(x) + c_{43}\pi_k, \end{aligned}$$

where  $c_{43} = (\frac{4}{5})^{N+2\alpha}$  and the last inequality follows by the fact of

$$|z-x| \leq |x| + |z| \leq 5/4 \quad \text{for } z \in B_{\frac{1}{4}}(0), x \in B_{\frac{1}{4}}(y_0).$$

Therefore,

$$\begin{aligned} (-\Delta)^\alpha \tilde{w}_k(x) + \tilde{w}_k^p(x) &\geq (-\Delta)^\alpha w_k(x) + w_k^p(x) + c_{43}\pi_k \\ &= c_{43}\pi_k, \quad x \in B_{\frac{1}{4}}(y_0). \end{aligned}$$

that is,  $\tilde{w}_k$  is a super solution of

$$\begin{aligned} (-\Delta)^\alpha u + u^p &= c_{43}\pi_k \quad \text{in } B_{\frac{1}{4}}(y_0), \\ u &= 0 \quad \text{in } B_{\frac{1}{4}}^c(y_0). \end{aligned} \tag{5.8}$$

Let  $\eta_1$  be the solution of

$$\begin{aligned} (-\Delta)^\alpha u &= 1 \quad \text{in } B_{\frac{1}{4}}(y_0), \\ u &= 0 \quad \text{in } B_{\frac{1}{4}}^c(y_0). \end{aligned}$$

Then  $(c_{43}\pi_k)^{\frac{1}{p}} \frac{\eta_1}{2 \max_{\mathbb{R}^N} \eta_1}$  is sub solution of (5.8) for  $k$  large enough. By Lemma 2.2, for  $k$  big we have

$$\tilde{w}_k(x) \geq (c_{43}\pi_k)^{\frac{1}{p}} \frac{\eta_1(x)}{2 \max_{\mathbb{R}^N} \eta_1}, \quad \forall x \in B_{\frac{1}{4}}(y_0),$$

which implies that

$$w_k(y_0) \geq \frac{(c_{43}\pi_k)^{\frac{1}{p}}}{2}.$$

Therefore, (5.7) and (5.5) imply

$$u_\infty(y_0) = \lim_{k \rightarrow \infty} u_k(y_0) \geq \lim_{k \rightarrow \infty} w_k(y_0) = \infty.$$

Since  $y_0$  is arbitrary on  $\partial B_1(0)$ , by (5.1), it follows that  $u_\infty = \infty$  in  $\mathbb{R}^N$ .  $\square$

**Proposition 5.3** *Suppose that  $p \in (1 + \frac{2\alpha}{N}, \frac{N}{N-2\alpha})$  and  $u_\infty$  is given by (1.15). Then  $u_\infty$  is a classical solution of (1.11).*

**Proof.** For  $p \in (1 + \frac{2\alpha}{N}, \frac{N}{N-2\alpha})$ , we observe that  $\tau_p := -\frac{2\alpha}{p-1} \in (-N, -N + 2\alpha)$ ,  $\tau_p - 2\alpha = \tau_p p$  and

$$(-\Delta)^\alpha |x|^{\tau_p} = c(\tau_p) |x|^{\tau_p - 2\alpha}, \quad x \in \mathbb{R}^N \setminus \{0\},$$

where  $c(\tau_p) < 0$ , see Lemma 3.1 in [19] and Lemma 3.2 in [19]. Let

$$W_p(x) = [-c(\tau_p)]^{\frac{1}{p-1}} |x|^{\tau_p}, \quad x \in \mathbb{R}^N \setminus \{0\}.$$

Then,  $W_p$  is a solution of (1.11).

*We first prove that*

$$u_\infty \leq W_p \quad \text{in } \mathbb{R}^N \setminus \{0\}. \tag{5.9}$$

In fact, we observe that  $u_k = \lim_{R \rightarrow \infty} u_{k,R}$  in  $\mathbb{R}^N \setminus \{0\}$ , where  $u_{k,R}$  is the solution of (4.1) with  $R > 1$  and

$$\lim_{x \rightarrow 0} u_{k,R}(x) |x|^{N-2\alpha} = c_{N,\alpha} k.$$

Then

$$\lim_{x \rightarrow 0} \frac{u_{k,R}(x)}{W_p(x)} = 0.$$

Moreover, we know that  $u_{k,R}$  is a classical solution of

$$\begin{aligned} (-\Delta)^\alpha u + u^p &= 0 \quad \text{in } B_R(0) \setminus \{0\}, \\ u &= 0 \quad \text{in } B_R^c(0). \end{aligned}$$

By Lemma 2.2 with  $O = B_R(0) \setminus B_\epsilon(0)$  and  $\epsilon > 0$  small enough, we obtain that

$$u_{k,R} \leq W_p \quad \text{in } \mathbb{R}^N \setminus \{0\},$$

which implies that for any  $k > 0$ ,

$$u_k \leq W_p \quad \text{in } \mathbb{R}^N \setminus \{0\}.$$

Thus, by the definition of  $u_\infty$ , (5.9) holds.

*Next we prove that  $u_\infty$  is a solution of (1.11).* We observe that  $W_p \in L^1(\mathbb{R}^N, d\omega)$ , then for any  $x_0 \neq 0$ , there exist  $c_{44}, c_{45} > 0$  independent of  $k$  such that

$$\|u_k\|_{L^1(\mathbb{R}^N, d\omega)} \leq c_{44} \quad \text{and} \quad \|u_k\|_{L^\infty(B_{\frac{|x_0|}{2}}(x_0))} \leq c_{45}.$$

It follows by the same argument in the proof of regularity in Lemma 4.2 that there exist  $\epsilon > 0$  and  $c_{46} > 0$  independent of  $k$  such that

$$\|u_k\|_{C^{2\alpha+\epsilon}(B_{\frac{|x_0|}{4}}(x_0))} \leq c_{46}.$$

By the definition of  $u_\infty$  and the Arzela-Ascoli Theorem, we find that  $u_\infty$  belongs to  $C^{2\alpha+\frac{\epsilon}{2}}(B_{\frac{|x_0|}{4}}(x_0))$ . Then,  $u_\infty$  is  $C^{2\alpha+\frac{\epsilon}{2}}$  locally in  $\mathbb{R}^N \setminus \{0\}$ . Since  $u_k$  is classical solution of (1.11), then it follows by Corollary 4.6 in [10] that  $u_\infty$  is a classical solution of (1.11).  $\square$

**Proof of Theorem 1.4.** The part (i) follows by Proposition 5.2. Now we prove (ii). By Proposition 5.3, we see that  $u_\infty$  is a classical solution of (1.11). It follows by uniqueness and rotation argument that  $u_k$  is radially symmetric. The definition of  $u_\infty$  and Proposition 5.1 yield  $u_\infty(x) = c_3 |x|^{-\frac{2\alpha}{p-1}}$ ,  $x \in \mathbb{R}^N \setminus \{0\}$ .  $\square$

## 5.2 Properties of $u_{\infty,\Omega}$

In this subsection, we make use of the properties of  $\{u_k\}$  and  $u_\infty$  to estimate the weak solution of (1.16) in general unbounded regular domain.

**Proof of Theorem 1.5.** First, for any  $k > 0$ , we use arguments in the proof of Lemma 4.2 to obtain that there exists a solution  $u_{k,\Omega}$  of (4.1) replaced  $B_R(0)$  by  $\Omega$  such that

$$u_k - m_{k,\Omega} \leq u_{k,\Omega} \leq u_k \quad \text{in } \Omega, \quad (5.10)$$

where  $m_{k,\Omega} = \sup_{x \in \Omega^c} u_k(x)$  and  $u_k$  is given by Theorem 1.2. Similarly to the proof of Proposition 4.1, we obtain that  $u_{k,\Omega}$  is unique and then

$$u_{k,\Omega} = \lim_{R \rightarrow \infty} u_{k,\Omega \cap B_R(0)}. \quad (5.11)$$

Since  $u_{k,\Omega \cap B_R(0)}$  is increasing respected to  $k$  and  $R$ , the mapping  $k \mapsto u_{k,\Omega}$  is increasing.

Next, by (5.11) we have that  $u_{\infty,\Omega} \geq u_{\infty,\Omega \cap B_R(0)}$  in  $\mathbb{R}^N$  for any  $R > 0$ . For  $p \in (0, \min\{1 + \frac{2\alpha}{N}, \frac{N}{2\alpha}\})$ , using [18, Theorem 1.1, Theorem 1.2], we have that  $\lim_{k \rightarrow \infty} u_{k,\Omega \cap B_R(0)} = \infty$  in  $\Omega \cap B_R(0)$ , then  $u_{\infty,\Omega} = \infty$  in  $\Omega \cap B_R(0)$  for any  $R > 0$ , which implies that

$$u_{\infty,\Omega} = \infty \quad \text{in } \Omega.$$

Finally, for  $p \in (1 + \frac{2\alpha}{N}, \frac{N}{N-2\alpha})$ , by (5.10) we have that

$$u_\infty - m_{\infty,\Omega} \leq u_{\infty,\Omega} \leq u_\infty \quad \text{in } \Omega,$$

where  $m_{\infty,\Omega} = \sup_{x \in \Omega^c} u_\infty(x) \geq m_{k,\Omega}$ , since  $\{u_k\}$  are increasing. Using arguments in the proof of Proposition 5.3, we obtain that  $u_{\infty,\Omega}$  is a classical solution of (1.17).  $\square$

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